

INTEGRABLE ISOTROPIC GEOMETRICAL FLOWS AND HEISENBERG FERROMAGNETS

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Abstract

Geometrical flows (GF) play an important role in modern mathematics and physics. In this letter we have considered some integrable isotropic GF – Ricci flows (RF) and mean curvature flows (MCF) – which are related with integrable Heisenberg ferromagnets. In 2+1 dimensions, these GF have a singularity at $t = t_0$.

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1 Introduction

Geometric flow (GF) is the gradient flow associated with a functional on a manifold which has a geometric interpretation, usually associated with some extrinsic or intrinsic curvature. A GF is also called a geometric evolution equation. They can be interpreted as flows on a moduli space (for intrinsic flows) or a parameter space (for extrinsic flows). These are of fundamental interest in the calculus of variations, and include several famous problems and theories. Particularly interesting are their critical points. GF play an important role in mathematics and physics.

In this note we will consider two examples of GF namely the Ricci flow (RF) and the mean curvature flow (MCF) related with the integrable Heisenberg ferromagnets (HF) in 1+1 and 2+1 dimensions. In particular we explore integrable reductions of the following (2+1)-dimensional GF:

$$\mathbf{r}_t = M\mathbf{n} - \xi + u\mathbf{r}_x, \quad (1.1a)$$

$$u_x = -\mathbf{r}_x \cdot (\mathbf{r}_{xx} \wedge \mathbf{r}_{xy}) \quad (1.1b)$$

and

$$\mathbf{r}_{xt} = M\mathbf{n} - \eta + u_x\mathbf{r}_{xy} + u_y\mathbf{r}_{xx}, \quad (1.2a)$$

$$u_{xx} - \alpha^2 u_{yy} = -2\alpha^2 \mathbf{r}_x \cdot (\mathbf{r}_{xx} \wedge \mathbf{r}_{xy}). \quad (1.2b)$$

Here $M(x, y, t)$ and $u(x, y, t)$ are scalar (real) functions. If $M = H$, then these GF can be considered as the examples of the (2+1)-dimensional MCF. The particular cases of these GF are the M-I flow and the Ishimori flow which as MCF are integrable.

2 Mean curvature flows

MCF is an example of a GF of hypersurfaces in a Riemannian manifold (for example, smooth surfaces in 3-dimensional Euclidean space).

2.1 Isotropic MCF

The isotropic MCF reads as (see, e.g. [1])

$$\mathbf{r}_t = H\mathbf{n} - \xi, \quad (2.1)$$

where $\mathbf{r} = (r_1, r_2, r_3)$, $\mathbf{n} = (n_1, n_2, n_3)$ $\xi = (\xi_1, \xi_2, \xi_3)$ and H is a mean curvature. Additionally in this note we assume that

$$\mathbf{r}_x^2 = 1. \quad (2.2)$$

In this note we use also the following form of MCF

$$\mathbf{r}_{xt} = H'\mathbf{n} + \eta, \quad (2.3)$$

where $\eta = (\eta_1, \eta_2, \eta_3)$.

2.2 Anisotropic MCF

There exist also the anisotropic MCF which can be written as

$$\mathbf{r}_t = H\mathbf{n} - \xi + \mathbf{V}, \quad (2.4a)$$

$$\mathbf{V}_x = \mathbf{r}_x \wedge J\mathbf{r}_x. \quad (2.4b)$$

This equation can be rewritten in the following equivalent form

$$\mathbf{r}_{xt} = H'\mathbf{n} + \eta + \mathbf{r}_x \wedge J\mathbf{r}_x, \quad (2.5)$$

where $J = \text{diag}(J_1, J_2, J_3)$, $\mathbf{V} = (V_1, V_2, V_3)$. Reference [4] presented examples of anisotropic MCF and RF.

3 Ricci flow

3.1 Isotropic RF

The RF is an intrinsic geometric flow—a process which deforms the metric of a Riemannian manifold—in this case in a manner formally analogous to the diffusion of heat, thereby smoothing out irregularities in the metric. It plays an important role in the proof of the Poincare conjecture.

Given a Riemannian manifold with metric tensor g_{ij} , we can compute the Ricci tensor R_{ij} , which collects averages of sectional curvatures into a kind of "trace" of the Riemann curvature tensor. If we consider the metric tensor (and the associated Ricci tensor) to be functions of a variable which is usually called "time" (but which may have nothing to do with any physical time), then the RF may be defined by the geometric evolution equation

$$g_{ijt} = -2R_{ij}. \quad (3.1a)$$

At the same time, the equation of the normalized RF reads

$$g_{ijt} = -2R_{ij} + \frac{2}{n} \langle R \rangle g_{ij}, \quad (3.1b)$$

where $\langle R \rangle$ is the average (mean) of the scalar curvature (which is obtained from the Ricci tensor by taking the trace) and n is the dimension of the manifold. Equation (3.2) preserves the volume of the metric.

3.2 Anisotropic RF

In the anisotropic case, Eqs. (3.1) take the form

$$g_{ijt} = -2R_{ij} + A_{ij} \quad (3.2a)$$

and

$$g_{ijt} = -2R_{ij} + \frac{2}{n} \langle R \rangle g_{ij} + A_{ij}, \quad (3.2b)$$

respectively.

4 HF flow

Consider the following HF

$$\mathbf{S}_y = \mathbf{S} \wedge \mathbf{S}_{xx}, \quad (4.1)$$

where $\mathbf{S} = (S_1, S_2, S_3)$, $\mathbf{S}^2 = 1$. If we assume

$$\mathbf{S} = \mathbf{r}_x, \quad (4.2)$$

then the equation (4.1) takes the form [5]

$$\mathbf{r}_{yx} = (\mathbf{r}_x \wedge \mathbf{r}_{xx})_x \quad (4.3a)$$

or

$$\mathbf{r}_y = \mathbf{r}_x \wedge \mathbf{r}_{xx}. \quad (4.3b)$$

The corresponding Lax representation is given by

$$\Phi_x = U\Phi, \quad \Phi_y = V\Phi, \quad (4.4)$$

where

$$U = \frac{\lambda}{2i}r_x, \quad V = \frac{i\lambda^2}{2}r_x + \frac{\lambda}{2}r_{xx}r_x, \quad r = \mathbf{r} \cdot \boldsymbol{\sigma}, \quad \boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3).$$

For the equation (4.3) we get

$$E = 1, \quad F = 0, \quad G = \mathbf{r}_y^2. \quad (4.5)$$

Here E, F, G and L, M, N are the coefficients of the fundamental forms of the surface

$$I = d\mathbf{r}^2 = g_{ij}dx^i dx^j = E dx^2 + 2F dx dy + G dy^2, \quad (4.6a)$$

$$II = d\mathbf{r} \cdot \mathbf{n} = b_{ij}dx^i dx^j = L dx^2 + 2M dx dy + N dy^2. \quad (4.6b)$$

In our case we have

$$R_{ij} = \frac{1}{2}Rg_{ij}, \quad (4.7a)$$

$$R = \frac{G_x^2 - 2GG_{xx}}{2G^2}, \quad (4.7b)$$

$$K = \frac{R}{2}. \quad (4.7c)$$

The modified RF related with the HF (4.1) can be written in the following form

$$g_{ijt} = -2R_{ij} + F_{ij}. \quad (4.8)$$

Now we present the MCF related with the HF equation (4.1). To do it, we consider the surfaces in R^3 associated to two parameters x and y and the renormalization group time t . It is convenient, where appropriate, to think of the surface as a graph of a function $r_3 = \varphi(r_1, (x, y; t), r_2(x, y; t); t)$ that evolves in time. In our case

$$r_{1y} = r_{2x}r_{3xx} - r_{xx}r_{3x}, \quad (4.9a)$$

$$r_{2y} = r_{3x}r_{1xx} - r_{3xx}r_{1x}, \quad (4.9b)$$

$$r_{3y} = r_{1x}r_{2xx} - r_{1xx}r_{2x}. \quad (4.9c)$$

In this system we must use the following expressions

$$r_{3y} = \varphi_{r_1}r_{1y} + \varphi_{r_2}r_{2y}, \quad (4.10a)$$

$$r_{3x} = \varphi_{r_1}r_{1x} + \varphi_{r_2}r_{2x}, \quad (4.10b)$$

$$r_{3xx} = \varphi_{r_1r_1}r_{1x}^2 + 2\varphi_{r_1}\varphi_{r_2}r_{1x}r_{2x} + \varphi_{r_2r_2}r_{2x}^2, \quad (4.10c)$$

From these and (2.2) we can define two functions r_{1x}, r_{1y} as

$$r_{1x} = \frac{-\varphi_{r_1}\varphi_{r_2}r_{2x} \pm \sqrt{1 + \varphi_{r_1}^2 - (1 + \varphi_{r_1}^2 + \varphi_{r_2}^2)r_{2x}^2}}{1 + \varphi_{r_1}^2}, \quad (4.11a)$$

$$r_{1y} = \frac{r_{1x}r_{2xx} - r_{1xx}r_{2x} - \varphi_{r_2}r_{2y}}{\varphi_{r_1}}. \quad (4.11b)$$

In this notation the mean curvature of the surface can be written as [1]

$$H = \frac{(1 + (\varphi_{r_2})^2)\varphi_{r_1r_1} + (1 + (\varphi_{r_1})^2)\varphi_{r_2r_2} - 2\varphi_{r_1}\varphi_{r_2}\varphi_{r_2r_2}}{(\sqrt{1 + (\varphi_{r_1})^2 + (\varphi_{r_2})^2})^3}. \quad (4.12)$$

The inward unit normal vector is given by

$$\mathbf{n} = \frac{1}{\sqrt{1 + (\varphi_{r_1})^2 + (\varphi_{r_2})^2}}(-\varphi_{r_1}, -\varphi_{r_2}, 1). \quad (4.13)$$

So for the MCF we have the following equation [1]

$$\begin{aligned} \varphi_t = & \frac{(1 + (\varphi_{r_2})^2)\varphi_{r_1r_1} + (1 + \varphi_{r_1}^2)\varphi_{r_2r_2}^2 - 2\varphi_{r_1}\varphi_{r_2}\varphi_{r_2r_2}}{1 + (\varphi_{r_1})^2 + (\varphi_{r_2})^2} + \\ & \xi_1\varphi_{r_1} + \xi_2\varphi_{r_2} - \xi_3. \end{aligned} \quad (4.14)$$

For the first and second fundamental forms of the two-dimensional surface and for the mean curvature we have the following system of equations [1]

$$g_{ijt} = -2HK_{ij}, \quad (4.15a)$$

$$g_t^{ij} = 2HK^{ij}. \quad 4.15b$$

Hence we get

$$(\ln \sqrt{g})_t = -H^2, \quad (4.16a)$$

$$K_{ijt} = g^{lm}\nabla_l\nabla_m K_{ij} - 2H(K^2)_{ij} + (Tr K^2)K_{ij}, \quad (4.16b)$$

$$\frac{\partial H}{\partial t} = g^{ij}\nabla_i\nabla_j H + (Tr K^2)H. \quad (4.16c)$$

5 M-I flow as MCF

Now we consider the following Myrzakulov I equation (abbreviated as the M-I equation)

$$\mathbf{S}_t = (\mathbf{S} \wedge \mathbf{S}_y + u\mathbf{S})_x, \quad (5.1a)$$

$$u_x = -\mathbf{S} \cdot (\mathbf{S}_x \wedge \mathbf{S}_y). \quad (5.1b)$$

After the identification (4.2) this system takes the form

$$\mathbf{r}_t = \mathbf{r}_x \wedge \mathbf{r}_{xy} + u\mathbf{r}_x, \quad (5.2a)$$

$$u_x = -\mathbf{r}_x \cdot (\mathbf{r}_{xx} \wedge \mathbf{r}_{xy}). \quad (5.2b)$$

This is M-I flow and is known to be integrable. The corresponding Lax representation is given by

$$\Phi_x = U\Phi, \quad (5.3a)$$

$$\Phi_t = \lambda\Phi_y + V\Phi, \quad (5.3b)$$

where

$$U = \frac{i\lambda}{2}r_x, \quad (5.4a)$$

$$V = \frac{\lambda}{4}([r_x, r_{xy}] + 2iur_x). \quad (5.4b)$$

Note that the M-I flow (5.2) can be written in the following form

$$\mathbf{r}_t = \left(\frac{MF}{\sqrt{g}} + u \right) \mathbf{r}_x - \frac{M}{\sqrt{g}} \mathbf{r}_y + \frac{G_x}{2\sqrt{g}} \mathbf{n} + \mathbf{c}, \quad (5.5a)$$

$$u_x = \frac{LG_x - 2MF_x}{2\sqrt{g}}. \quad (5.5b)$$

Here $\mathbf{c} = \mathbf{c}(y, t)$ which we set to $\mathbf{c} = 0$. For the M-I flow we have

$$g_{11t} = E_t = 0, \quad (5.6a)$$

$$g_{12t} = F_t = \frac{(FM - N)F_x}{\sqrt{g}} - L_y\sqrt{g} + uF_x + u_y, \quad (5.6b)$$

$$g_{22t} = G_t = \frac{(FM - N)G_x}{\sqrt{g}} - 2M_y\sqrt{g} + uG_x + 2Fu_y. \quad (5.6c)$$

So that for $g = \det(g_{ij})$ we get

$$g_t = G_t - 2FF_t = \frac{g_x}{\sqrt{g}}(FM - N) - 2\sqrt{g}(M_y - FL_y) + ug_x. \quad (5.7)$$

Equations (5.6)-(5.7) are integrable. Note that in our case we have the following formula

$$R_{ij} = \frac{1}{2}Rg_{ij} \quad (5.8)$$

and

$$R = \frac{1}{2g^2} (4FF_xF_y - 2F_xG_y - 2FF_xG_x + G_x^2 - 4F^2F_{xy} + 4GF_{xy} + 2F^2G_{xx} - 2GG_{xx}). \quad (5.9)$$

As in the previous case it is convenient, where appropriate, to think of the surface as the graph of a function $r_3 = \varphi(r_1(x, y; t), r_2(x, y; t); t)$ that evolves in time. First let us rewrite the system (5.2) in component form. We have

$$r_{1t} = r_{2x}r_{3xy} - r_{2xy}r_{3x} + ur_{1x}, \quad (5.10a)$$

$$r_{2t} = r_{3x}r_{1xy} - r_{3xy}r_{2x} + ur_{2x}, \quad (5.10b)$$

$$r_{3t} = r_{1x}r_{2xy} - r_{1xy}r_{2x} + ur_{3x}, \quad (5.10c)$$

$$u_x = -r_{1xx}(r_{2xx}r_{3xy} - r_{2xy}r_{3xx}) - r_{2xx}(r_{3xx}r_{1xy} - r_{3xy}r_{1xx}) - r_{3xx}(r_{1xx}r_{2xy} - r_{1xy}r_{2xx}). \quad (5.10d)$$

As

$$r_{3y} = \varphi_{r_1}r_{2y} + \varphi_{r_2}r_{2y}, \quad (5.11a)$$

$$r_{3x} = \varphi_{r_1}r_{1x} + \varphi_{r_2}r_{2x}, \quad (5.11b)$$

$$r_{3xx} = \varphi_{r_1r_1}r_{1x}^2 + 2\varphi_{r_1}\varphi_{r_2}r_{1x}r_{2x} + \varphi_{r_2r_2}r_{2x}^2, \quad (5.11c)$$

for the functions r_{1x}, r_{1y} we get

$$r_{1x} = \frac{-\varphi_{r_1}\varphi_{r_2}r_{2x} \pm \sqrt{1 + \varphi_{r_1}^2 - (1 + \varphi_{r_1}^2 + \varphi_{r_2}^2)r_{2x}^2}}{1 + \varphi_{r_1}^2}, \quad (5.12a)$$

$$r_{1y} = \frac{r_{1x}r_{2xx} - r_{1xx}r_{2x} - \varphi_{r_2}r_{2y}}{\varphi_{r_1}}. \quad (5.12b)$$

So for the mean curvature of the surface we have the following formula

$$H = \frac{(1 + (\varphi_{r_2})^2)\varphi_{r_1}^2 + (1 + \varphi_{r_1}^2)\varphi_{r_2}^2 - 2\varphi_{r_1}\varphi_{r_2}\varphi_{r_1r_2}}{(\sqrt{1 + \varphi_{r_1}^2 + \varphi_{r_2}^2})^3}. \quad (5.13)$$

On the other hand the inward unit normal vector is defined by the formula [1]

$$\mathbf{n} = \frac{1}{\sqrt{1 + \varphi_{r_1}^2 + \varphi_{r_2}^2}}(-\varphi_{r_1}, -\varphi_{r_2}, 1). \quad (5.14)$$

Now we are ready to find the MCF. As result we obtain

$$\begin{aligned} \varphi_t = & \frac{(1 + (\varphi_{r_2})^2)\varphi_{r_1r_1} + (1 + (\varphi_{r_1})^2)\varphi_{r_2r_2} - 2\varphi_{r_1}\varphi_{r_2}\varphi_{r_2r_2}}{1 + (\varphi_{r_1})^2 + (\varphi_{r_2})^2} + \\ & \xi_1\varphi_{r_1} + \xi_2\varphi_{r_2} - \xi_3. \end{aligned} \quad (5.15)$$

Finally we present the following two equations [1]

$$(\ln \sqrt{g})_{tt} = -2Hg^{ij}\nabla_i\nabla_jH - 2(TrK^2)H^2 \quad (5.16)$$

and

$$(\sqrt{g})_{tt} = [-2Hg^{ij}\nabla_i\nabla_jH - 2(TrK^2)H^2 + H^4]\sqrt{g}. \quad (5.17)$$

6 M-I flow as RF

Consider a Riemannian manifold M of dimension 3 with a local coordinate system X^μ and metric $G_{\mu\nu}(X)$ so that its first fundamental form is

$$ds_M^2 = G_{\mu\nu}(X) dX^\mu dX^\nu \quad (6.1a)$$

or

$$ds^2 = G_{11}dx^2 + G_{22}dy^2 + G_{33}dt^2 + 2G_{12}dxdy + 2G_{13}dxdt + 2G_{23}dydt. \quad (6.1b)$$

Here $X^1 = x$, $X^2 = y$, $X^3 = t$ and

$$\begin{aligned} G_{11} &= \mathbf{r}_x^2, & G_{22} &= \mathbf{r}_y^2, & G_{33} &= \mathbf{r}_t^2, \\ G_{12} &= G_{21} = \mathbf{r}_x \cdot \mathbf{r}_y, & G_{13} &= G_{31} = \mathbf{r}_x \cdot \mathbf{r}_t, & G_{23} &= G_{32} = \mathbf{r}_y \cdot \mathbf{r}_t. \end{aligned} \quad (6.2)$$

For the M-I flow the metric takes the form

$$\begin{aligned} G_{11} &= \mathbf{r}_x^2 = 1, & G_{22} &= \mathbf{r}_y^2, & G_{33} &= \mathbf{r}_{xy}^2 + u, \\ G_{12} &= G_{21} = \mathbf{r}_x \cdot \mathbf{r}_y, & G_{13} &= G_{31} = u, & G_{23} &= G_{32} = u\mathbf{r}_y \cdot \mathbf{r}_y - \mathbf{r}_{xy} \cdot (\mathbf{r}_x \wedge \mathbf{r}_y). \end{aligned} \quad (6.3)$$

In this case $G = \det(G_{ij})$ is given by

$$G = u^2[\mathbf{r}_{xy}^2 - \mathbf{r}_y^2] + u[\mathbf{r}_y^2 - (\mathbf{r}_x \cdot \mathbf{r}_y)^2] + [\mathbf{r}_y^2 - (\mathbf{r}_x \cdot \mathbf{r}_y)^2]\mathbf{r}_{xy}^2. \quad (6.4)$$

For the M-I flow the corresponding modified RF reads as

$$G_{ijt} = -2R_{ij} + F_{ij}. \quad (6.5)$$

This modified RF is integrable. This follows from the integrability of the original equation (5.2).

7 Conclusion

In this letter we have considered some integrable and non integrable GF related with some integrable HF in 1+1 and 2+1.

Also we would like to note that the flows considered in this letter have a singularity with respect to t . This is related with the following fact: The spectral parameter λ in the Lax representation (5.3) obeys the following nonlinear equation

$$\lambda_t = \lambda\lambda_y. \quad (7.1)$$

This equation has the following solution

$$\lambda = \frac{a + y}{t_0 - t}. \quad (7.2)$$

It has a singularity at $t = t_0$. This means that RF and MCF, which are related with the M-I flow, also have a singularity at this point. Also we note that references [2]-[3]

considered GF related to the Ishimori equation. Finally we would like to present the following systems [3]

$$g_{ijt} = -2R_{ij} + ug_{ij}, \quad (7.3a)$$

$$u_{tt} = \Delta(u + kR) \quad (7.3b)$$

and

$$g_{ijt} = -2R_{ij} + ug_{ij}, \quad (7.4a)$$

$$u_t = \Delta(u + kR). \quad (7.4b)$$

More general forms of these systems look like [3]

$$g_{ijt} = -2R_{ij} + (\beta u + \alpha R)g_{ij}, \quad (7.5a)$$

$$u_{tt} = \Delta(u + kR) \quad (7.5b)$$

and

$$g_{ijt} = -2R_{ij} + (\beta u + \alpha R)g_{ij}, \quad (7.6a)$$

$$u_t = \Delta(u + kR), \quad (7.6b)$$

respectively.

References

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